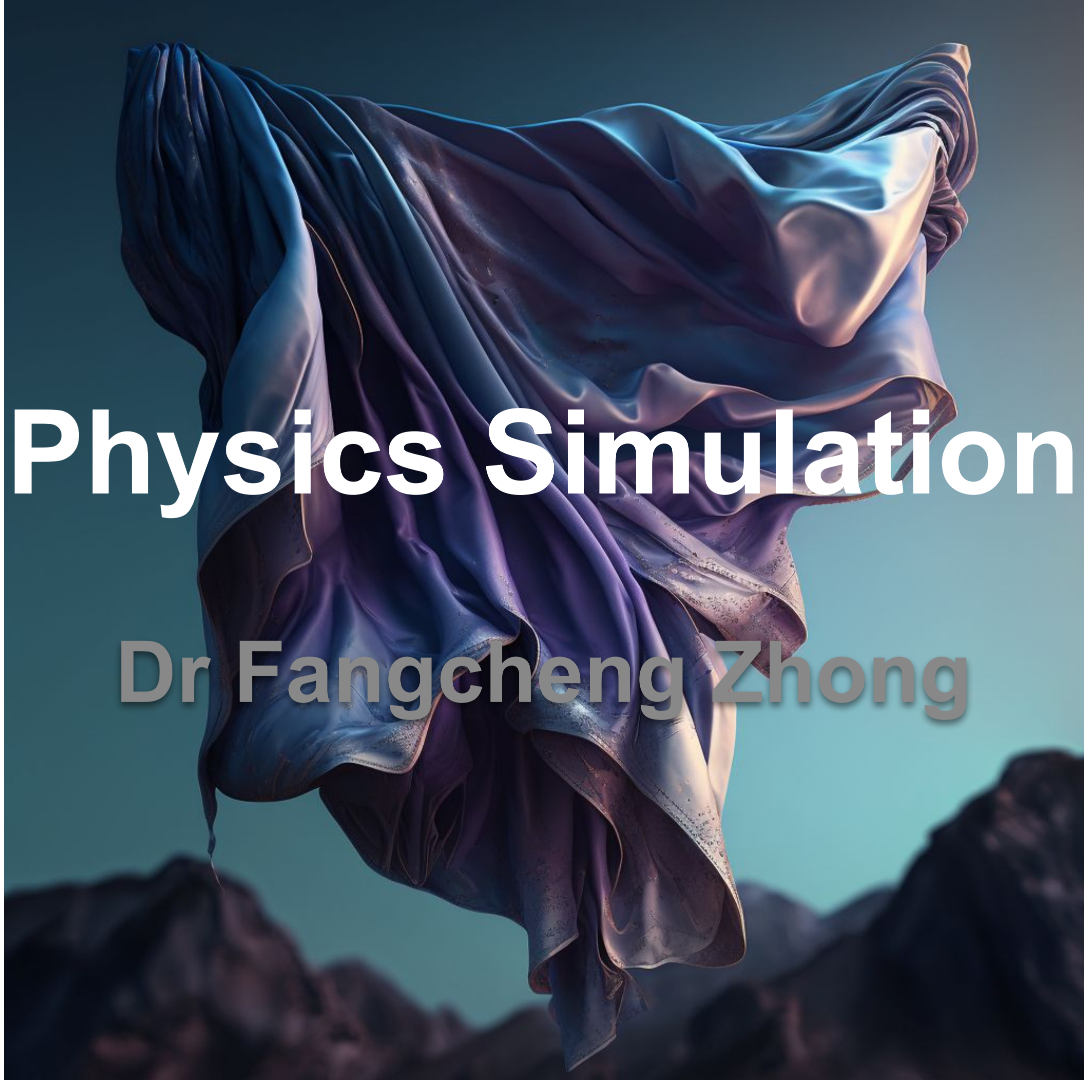
Physics Simulation

Dr Fangcheng Zhong



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Outline

- Physically-based simulation
  - Particle system
  - Rigid-body dynamics
  - Mass-spring system
  - Fluid dynamics
- Numerical solvers for differential equations
  - Finite difference methods
    - Euler method, Runge-Kutta method, trapezoidal rule
    - truncation error, convergence, stability

Particle System

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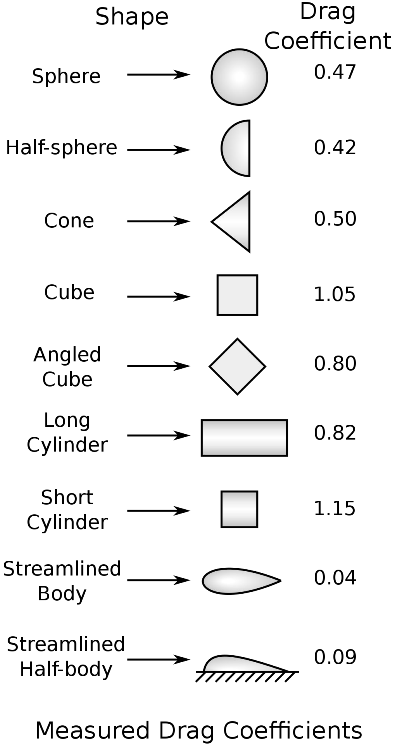
Particle System

initialisation

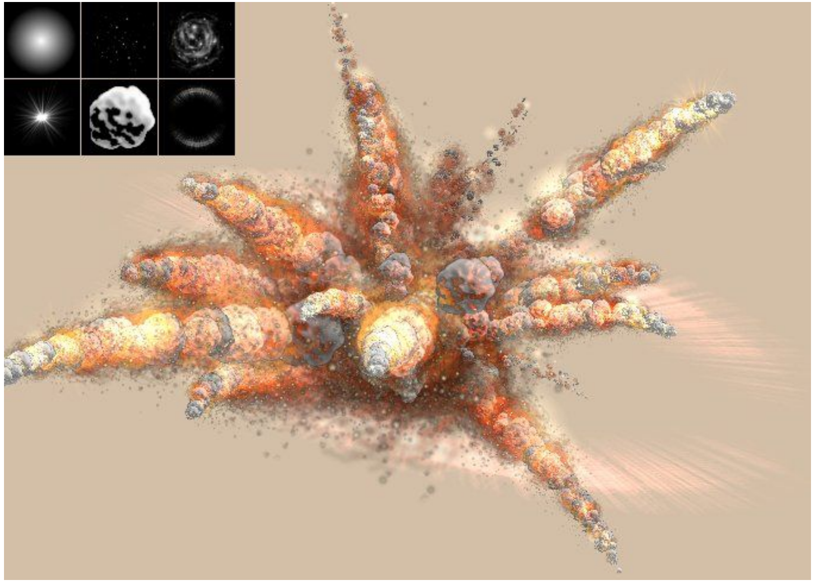
$$\mathbf{x}(0),\ \mathbf{x}'(0)$$

simulation

$$\mathbf{f} = m\mathbf{a} = m\mathbf{x}''$$



rendering



Rigid Body Dynamics

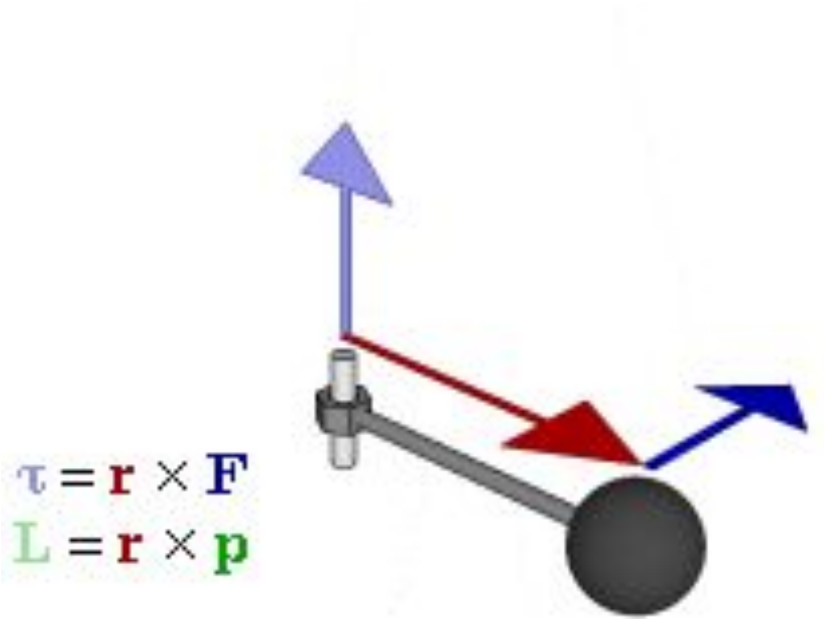
Rigid Body Dynamics

linear acceleration

$$\mathbf{f} = m\mathbf{a} = m\mathbf{x}''$$

angular acceleration

$$\boldsymbol{\tau} = mr^2\boldsymbol{\alpha} = mr^2\boldsymbol{\omega}'$$



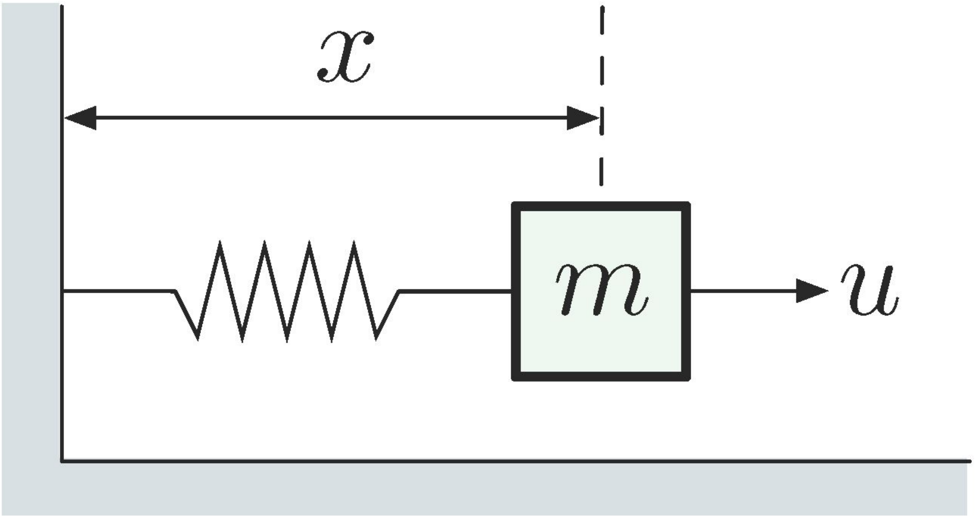
Deformation

elastic

plastic

both

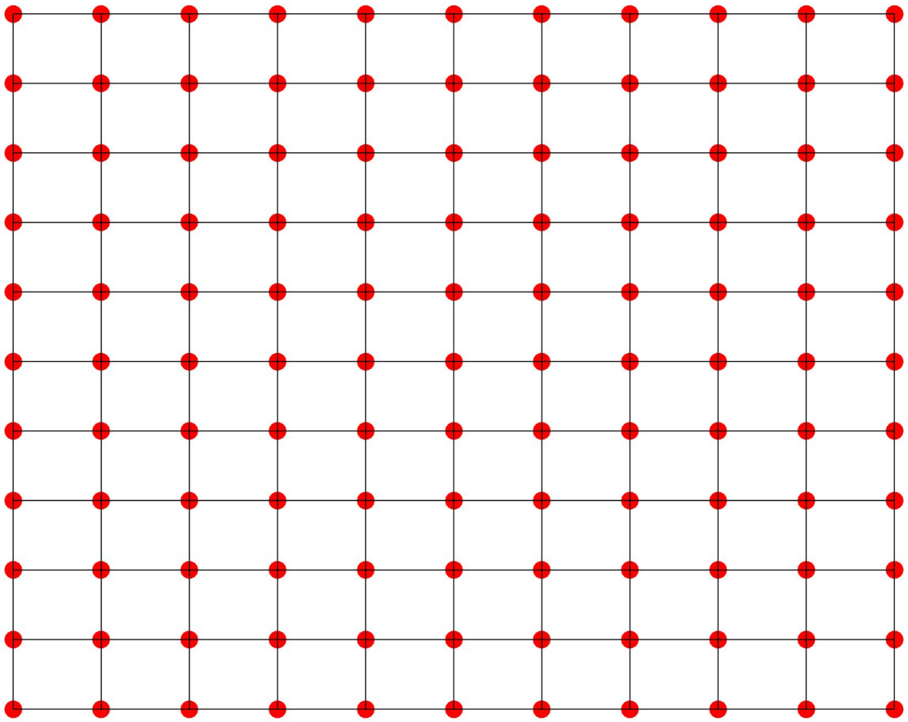
Mass-Spring System



Hooke's law

$$f = -k\left(\|x\| - l\right) = m x''$$

Mass-Spring System



cloth simulation

WebGL demo (link1) (link2)

Fluid Dynamics

Lagrangian
vs
Eulerian
specification

Finite Difference Methods

- Approximate derivatives with finite differences to solve differential equations

$$F(t, x(t), x'(t), \ldots, x^{(n)}(t)) = 0$$

general-form ordinary differential equation (ODE)

Truncation Error

Taylor expansion

$$f(x) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f'''(a)}{3!}h^3 + \cdots \frac{f^{(n)}(a)}{n!}h^n + \cdots$$

h = (x - a)

$$\underbrace{\frac{f(a+h) - f(a)}{h}}_{\text{1st order approximation}} = f'(a) + \underbrace{\frac{f''(a)}{2!}h + \frac{f'''(a)}{3!}h^2 + \cdots \frac{f^{(n)}(a)}{n!}h^{n-1} + \cdots}_{\text{truncation error}}$$

Finite Difference Methods

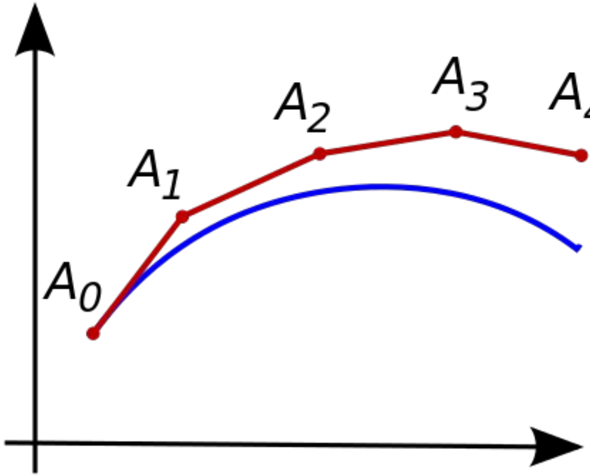
finite difference coefficients

Derivative	Accuracy	−4	−3	−2	−1	0	1	2	3	4
1	2				−1/2	0	1/2			
	4			1/12	−2/3	0	2/3	−1/12		
	6		−1/60	3/20	−3/4	0	3/4	−3/20	1/60	
	8	1/280	−4/105	1/5	−4/5	0	4/5	−1/5	4/105	−1/280
2	2				1	−2	1			
	4			−1/12	4/3	−5/2	4/3	−1/12		
	6		1/90	−3/20	3/2	−49/18	3/2	−3/20	1/90	
	8	−1/560	8/315	−1/5	8/5	−205/72	8/5	−1/5	8/315	−1/560
3	2			−1/2	1	0	−1	1/2		
	4		1/8	−1	13/8	0	−13/8	1	−1/8	
	6	−7/240	3/10	−169/120	61/30	0	−61/30	169/120	−3/10	7/240
4	2			1	−4	6	−4	1		
	4		−1/6	2	−13/2	28/3	−13/2	2	−1/6	
	6	7/240	−2/5	169/60	−122/15	91/8	−122/15	169/60	−2/5	7/240

Euler Method

$$X'(t) = f(X(t), t)$$

$$X(t_{n+1}) = X(t_n) + \Delta t f(X(t_n), t_n)$$



- 1st order explicit method

Runge–Kutta methods

RK2

$$X(t_{n+1/2}) = X(t_n) + \frac{1}{2}\Delta t f(X(t_n), t_n)$$

$$X(t_{n+1}) = X(t_n) + \Delta t f(X(t_{n+1/2}), t_{n+1/2})$$

- 2nd order explicit method

Trapezoidal Rule

$$X(t_{n+1}) = X(t_n) + \Delta t(\frac{1}{2}f(X(t_n), t_n) + \frac{1}{2}f(X(t_{n+1}), t_{n+1}))$$

- 2nd order implicit method

Finite Difference Methods

- convergence/accuracy
- stability
- computational efficiency
- memory efficiency

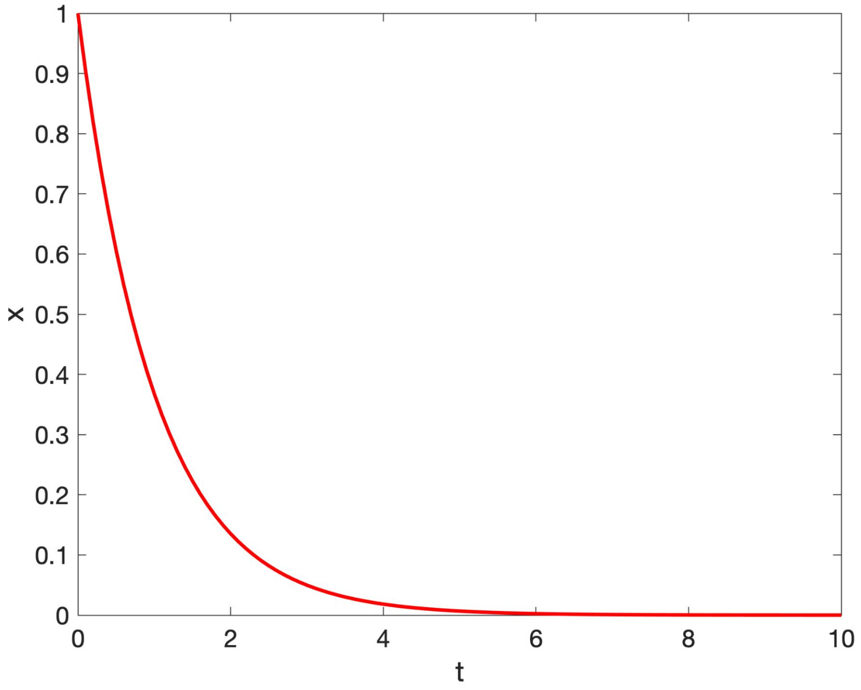
A-Stability

test equation

$$x' = kx$$

$$x = e^{kt}$$

$$x \to 0 \text{ as } t \to \infty \text{ if } \Re(k) < 0$$



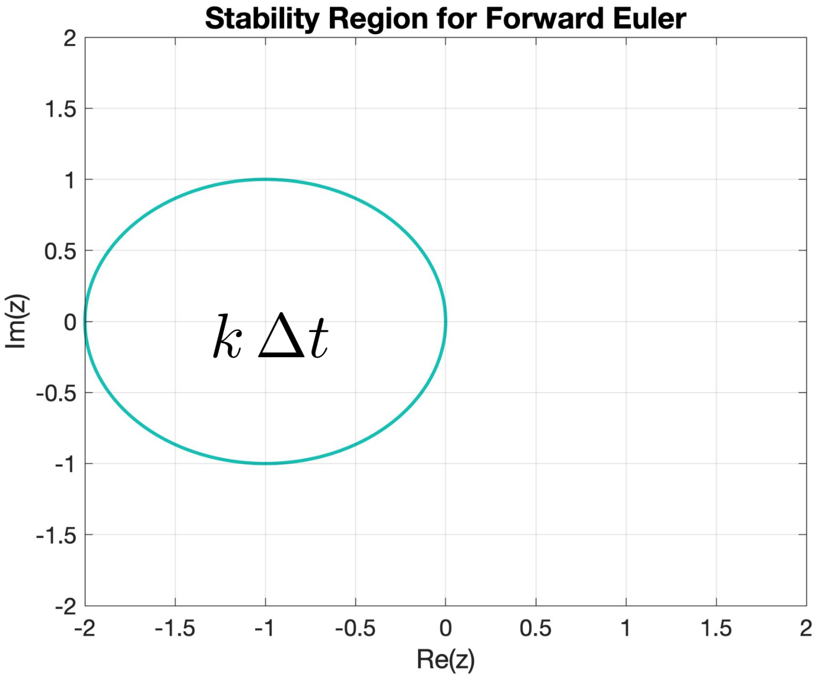
A-Stability

Forward Euler

$$x(t_{n+1}) = x(t_n) + k\,x(t_n)\Delta t$$

$$x(t_{n+1}) = (1 + k\,\Delta t)\,x(t_n)$$

$$x(t_n) = \underbrace{(1 + k\,\Delta t)^n}_{\text{stable when} < 1}\ x(t_0)$$



Higher Order ODEs

2nd order ODE $$x''(t) = f(t, x(t), x'(t))$$

reparameterization $$x'(t) = v(t)$$
$$v'(t) = f(t, x(t), v(t))$$

Euler method $$x(t_{n+1}) = x(t_n) + \Delta t\, v(t_n)$$
$$v(t_{n+1}) = v(t_n) + \Delta t\, f(t_n, x(t_n), v(t_n))$$